

# Lawvere-Tierney sheaves, factorization systems, sections and $j$ -essential monomorphisms in a topos

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## Abstract

Let  $j$  be a Lawvere-Tierney topology (a topology, for short) on an arbitrary topos  $\mathcal{E}$ ,  $B$  an object of  $\mathcal{E}$ , and  $j_B = j \times 1_B$  the induced topology on the slice topos  $\mathcal{E}/B$ . In this manuscript, we analyze some properties of the pullback functor  $\Pi_B : \mathcal{E} \rightarrow \mathcal{E}/B$  which have deal with topology. Then for a left cancelable class  $\mathcal{M}$  of all  $j$ -dense monomorphisms in a topos  $\mathcal{E}$ , we achieve some necessary and sufficient conditions for that  $(\mathcal{M}, \mathcal{M}^\perp)$  is a factorization system in  $\mathcal{E}$ , which is related to the factorization systems in slice topoi  $\mathcal{E}/B$ , where  $B$  ranges over the class of objects of  $\mathcal{E}$ . Among other things, we prove that an arrow  $f : X \rightarrow B$  in  $\mathcal{E}$  is a  $j_B$ -sheaf whenever the graph of  $f$ , is a section in  $\mathcal{E}/B$  as

well as the object of sections  $S(f)$  of  $f$ , is a  $j$ -sheaf in  $\mathcal{E}$ . Furthermore, we introduce a class of monomorphisms in  $\mathcal{E}$ , which we call them  $j$ -essential. Some equivalent forms of those and some of their properties are presented. Also, we prove that any presheaf in a presheaf topos has a maximal essential extension. Finally, some similarities and differences of the obtained result are discussed if we put a (productive) weak topology  $j$ , studied by some authors, instead of a topology.

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## 1 Introduction and background

A Lawvere-Tierney topology is a logical connective for modal logic. Recently, applications of Lawvere-Tierney topologies in broad topics such as measure theory [7] and quantum Physics [14, 15] are observed. In the spacial case, considerable work has been presented that is dedicated to the study of (weak) Lawvere-Tierney topology on a presheaf topos on a small category and especially on a monoid, see [6, 5]. It is clear that Lawvere-Tierney sheaves in a topos are exactly injective objects (of course, with respect to dense monomorphisms, not to merely monomorphisms) which are separated too. Injectivity with respect to a class  $\mathcal{M}$  of morphisms in a slice category  $\mathcal{C}/B$  (which its objects are  $\mathcal{C}$ -arrows with codomain  $B$ ) has been studied in extensive form, for example we refer the reader to [1, 3]. From this perspective, in this paper we will establish some categorical characterizations of injectives in slice topoi to sheaves. The object of sections  $S(f)$  of  $f$  is a notion which in [3] it is related to injective objects in a slice category. This object is very useful in synthetic differential geometry (or SDG, for short) (for details, see [11]). For example, considering  $D$  as infinitesimals, for any micro-linear object  $M$  we have:

- Let  $\tau$  be the tangent bundle on  $M$ , i.e.,  $\tau : M^D \rightarrow M$ , which is defined by  $\tau(t) = t(0)$ . Then  $S(\tau)$  is all vector fields on  $M$ .

- Consider  $\eta : M^{D \times D} \rightarrow M$  which assigns to any micro-square  $Q$  of  $M^{D \times D}$ , the element  $Q(0,0)$ . Then,  $S(\eta)$  is all distributions of dimension 2 on  $M$ .

Throughout this paper,  $\mathcal{E}$  is a (elementary) topos, two objects  $0, 1$  are the initial and terminal objects and the object  $\Omega$  together with the arrow  $1 \xrightarrow{\text{true}} \Omega$  is the subobject classifier of  $\mathcal{E}$ . Also, the arrow  $\wedge : \Omega \times \Omega \rightarrow \Omega$  is the meet operation on  $\Omega$ . Now, we express some basic concepts from [12] which will be needed in sequel.

**Definition 1.1.** A *Lawvere-Tierney topology* on  $\mathcal{E}$  is a map  $j : \Omega \rightarrow \Omega$  in  $\mathcal{E}$  satisfies the following properties

- (a)  $j \circ \text{true} = \text{true}$ ; (b)  $j \circ j = j$ ; (c)  $j \circ \wedge = \wedge \circ (j \times j)$ ;

$$\begin{array}{ccc}
 1 & \xrightarrow{\text{true}} & \Omega \\
 & \searrow \text{true} & \downarrow j \\
 & & \Omega
 \end{array}
 \quad
 \begin{array}{ccc}
 \Omega & \xrightarrow{j} & \Omega \\
 & \searrow j & \downarrow j \\
 & & \Omega
 \end{array}
 \quad
 \begin{array}{ccc}
 \Omega \times \Omega & \xrightarrow{\wedge} & \Omega \\
 j \times j \downarrow & & \downarrow j \\
 \Omega \times \Omega & \xrightarrow{\wedge} & \Omega
 \end{array}$$

Form now on, we say briefly to a Lawvere-Tierney topology on  $\mathcal{E}$ , a *topology* on  $\mathcal{E}$ .

Recall [12] that topologies on  $\mathcal{E}$  are in one to one correspondence with universal closure operators. For a topology  $j$  on  $\mathcal{E}$ , considering  $\overline{(\cdot)}$  as the universal closure operator corresponding to  $j$ , a monomorphism  $k : A \rightarrowtail C$  in  $\mathcal{E}$  is called  *$j$ -dense* whenever  $\overline{A} = C$ , as two subobjects of  $C$ . Also, we say that  $k$  is  *$j$ -closed* if we have  $\overline{A} = A$ , again as subobjects of  $C$ .

**Definition 1.2.** For a topology  $j$  on  $\mathcal{E}$ , an object  $F$  of  $\mathcal{E}$  is called a  *$j$ -sheaf* whenever for any  $j$ -dense monomorphism  $m : A \rightarrowtail E$ , one can uniquely extend any arrow  $h : A \rightarrow F$  to a map  $g$  on all of  $E$ ,

$$\begin{array}{ccc}
 A & \xrightarrow{h} & F \\
 m \downarrow & \nearrow g & \\
 E & & 
 \end{array}
 \tag{1}$$

We say that  $F$  is  *$j$ -separated* if the arrow  $g$  exists in (1), it is unique.

We will denote the full subcategories of  $\mathcal{E}$  consisting of  $j$ -sheaves and  $j$ -separated objects as  $\mathbf{Sh}_j(\mathcal{E})$  and  $\mathbf{Sep}_j(\mathcal{E})$ , respectively.

We now briefly describe the contents of other sections. We start in Section 2, to study basic properties of the pullback functor  $\Pi_B : \mathcal{E} \rightarrow \mathcal{E}/B$ , for any object  $B$  of  $\mathcal{E}$ , along with the unique map  $!_B : B \rightarrow 1$ . Afterwards, we would like to achieve, for a left cancelable class  $\mathcal{M}$  of all  $j$ -dense monomorphisms in a topos  $\mathcal{E}$ , some necessary and sufficient conditions for that  $(\mathcal{M}, \mathcal{M}^\perp)$  to be a factorization system in  $\mathcal{E}$ , which is related to the factorization systems in slice topoi  $\mathcal{E}/B$ . In section 3, among other things, we prove that an arrow  $f : X \rightarrow B$  in  $\mathcal{E}$  is a  $j_B$ -sheaf whenever the graph of  $f$ , is a section in  $\mathcal{E}/B$  as well as the object of sections  $S(f)$  of  $f$ , is a  $j$ -sheaf in  $\mathcal{E}$ . In section 4, we introduce a class of monomorphisms in an elementary topos  $\mathcal{E}$ , which we call them ‘ $j$ -essential monomorphisms’. We present some equivalent forms of these and some of their properties. Meanwhile, we prove that any presheaf in a presheaf topos has a maximal essential extension. It is shown that the functor  $\Pi_B$  reflects  $j$ -essential extensions. It is seen that some of these results hold for a (*productive*) *weak topology*  $j$ , studied in [10], instead of a topology as well.

## 2 Pullback functors, left cancelable dense monomorphisms and factorization systems

The purpose of this section is to present some basic properties of the pullback functor  $\Pi_B : \mathcal{E} \rightarrow \mathcal{E}/B$ , for any object  $B$  of  $\mathcal{E}$ , along with the unique map  $!_B : B \rightarrow 1$ . Afterwards, for a left cancelable class  $\mathcal{M}$  of all  $j$ -dense monomorphisms in a topos  $\mathcal{E}$  we achieve some necessary and sufficient conditions for that  $(\mathcal{M}, \mathcal{M}^\perp)$  to be a factorization system in  $\mathcal{E}$ , which is related to the factorization systems in slice topoi  $\mathcal{E}/B$ .

To begin with, the following lemma characterizes sheaves in a topos  $\mathcal{E}$ .

**Lemma 2.1.** *Let  $j$  be a topology on  $\mathcal{E}$ . Then an object  $E$  of  $\mathcal{E}$  is  $j$ -sheaf iff  $E$  is  $j$ -unique absolute retract; that is, any  $j$ -dense monomorphism  $u : E \rightarrowtail F$ , has a unique retraction  $v : F \rightarrow E$ .*

**Proof.** *Necessity.* Since  $E$  is a  $j$ -sheaf, for any  $j$ -dense monomorphism  $u : E \rightarrowtail F$ , corresponding to the identity map  $\text{id}_E : E \rightarrow E$  there exists a unique map  $v : F \rightarrow E$  such that the following diagram commutes.

$$\begin{array}{ccc} E & \xrightarrow{\text{id}_E} & E \\ u \downarrow & \nearrow v & \\ F & & \end{array}$$

*Sufficiency.* For each  $j$ -dense monomorphism  $m : U \rightarrowtail V$  and any map  $f : U \rightarrow E$ , we construct the following pushout diagram in  $\mathcal{E}$ .

$$\begin{array}{ccc} U & \xrightarrow{f} & E \\ m \downarrow & & \downarrow n \\ V & \xrightarrow[p.o.]{g} & F \end{array} \quad (2)$$

Since in any topos pushouts transfer  $j$ -dense monomorphisms (see [9]), so, in (2),  $n$  is  $j$ -dense and hence by assumption, there exists a unique retraction  $p : F \rightarrow E$  such that  $pn = \text{id}_E$ . Now, for the the arrow  $pg : V \rightarrow E$  we have  $pgm = pnf = \text{id}_E f = f$ . To prove that  $pg : V \rightarrow E$  with this property is unique, let  $h : V \rightarrow E$  be an arrow in  $\mathcal{E}$  in such a way that  $hm = f$ . Then, in the pushout diagram (2), according to the maps  $h : V \rightarrow E$  and  $\text{id}_E : E \rightarrow E$ , there exists a unique map  $k : F \rightarrow E$  such that  $kn = \text{id}_E$  and  $kg = h$ .

$$\begin{array}{ccc} U & \xrightarrow{f} & E \\ m \downarrow & & \downarrow n \\ V & \xrightarrow{g} & F \end{array} \quad \begin{array}{c} \text{id}_E \\ \nearrow k \\ \searrow h \end{array}$$

Now,  $k$  is a retraction of  $j$ -dense monomorphism  $n$ , so by hypothesis we get  $p = k$ . Consequently,  $pg = kg = h$ .  $\square$

For an object  $B$  of  $\mathcal{E}$ , we consider the pullback functor  $\Pi_B : \mathcal{E} \rightarrow \mathcal{E}/B$  along with the unique map  $!_B : B \rightarrow 1$ , which assigns to any  $A$  of  $\mathcal{E}$ , the second projection  $\Pi_B(A) = \pi_B^A : A \times B \rightarrow B$  and to any  $f : A \rightarrow C$ , the arrow  $f \times \text{id}_B : A \times B \rightarrow C \times B$  in  $\mathcal{E}$  such that  $\pi_B^C(f \times \text{id}_B) = \pi_B^A$ . Recall [12] that the object  $\pi_B^\Omega$  together with the

arrow

$$\text{true} \times \text{id}_B : \text{id}_B \longrightarrow \pi_B^\Omega$$

is the subobject classifier of the slice topos  $\mathcal{E}/B$ . Also, in a similar vein, we can observe that the meet operation  $\wedge_B$  on  $\pi_B^\Omega$  is the arrow  $\wedge \times 1_B$  in  $\mathcal{E}$  such that  $\pi_B^\Omega(\wedge \times 1_B) = \pi_B^{\Omega \times \Omega}$ ,

$$\begin{array}{ccc} \Omega \times \Omega \times B & \xrightarrow{\wedge \times 1_B} & \Omega \times B \\ & \searrow \pi_B^{\Omega \times \Omega} & \downarrow \pi_B^\Omega \\ & & B. \end{array}$$

Now, by Definition 1.1, we easily get the following lemma.

**Lemma 2.2.** *Let  $B$  be any object in a topos  $\mathcal{E}$ . Then any topology  $k : \pi_B^\Omega \rightarrow \pi_B^\Omega$  on  $\mathcal{E}/B$  is a pair  $(l, \pi_B^\Omega)$ , for some arrow  $l : \Omega \times B \rightarrow \Omega$  in  $\mathcal{E}$  satisfies the following conditions (as arrows in  $\mathcal{E}$ )*

- (1)  $l \circ (l, \pi_B^\Omega) = l$ ;
- (2)  $l \circ (\text{true} \times 1_B) = \text{true} \circ !_B$ ;
- (3)  $l \circ \wedge_B = \wedge \circ (l \circ (\pi_1, \pi_3), l \circ (\pi_2, \pi_3))$ , where  $\pi_i$  is the  $i$ -th projection on  $\Omega \times \Omega \times B$ , for  $i = 1, 2, 3$ .

By Lemma 2.2, for each topology  $j$  on  $\mathcal{E}$ , considering  $l = j \circ \pi_\Omega^B$ , it is easily seen that  $j \times 1_B = (l, \pi_B^\Omega)$  is a topology on  $\mathcal{E}/B$  which we denote it by  $j_B$ . In this case  $j_B$  is called the *induced topology* on  $\mathcal{E}/B$  by  $j$ .

One can simply see that if an arrow  $k$  is a monomorphism in  $\mathcal{E}/B$ , then  $k$  as an arrow in  $\mathcal{E}$ , is too. Also, for each monomorphism  $k : f \rightarrowtail g$  in  $\mathcal{E}/B$ , where  $f : X \rightarrow B$  and  $g : Y \rightarrow B$  in  $\mathcal{E}$ , we can observe

$$\widetilde{f} \xrightarrow{\widetilde{k}} g = (\overline{X} \xrightarrow{g\overline{k}} B) \xrightarrow{\overline{k}} g, \quad (3)$$

where  $\overline{(\cdot)}$  and  $\widetilde{(\cdot)}$  are the universal closure operators corresponding to  $j$  and  $j_B$  on topoi  $\mathcal{E}$  and  $\mathcal{E}/B$ , respectively, in which whole and the middle squares of the following diagram are pullbacks in  $\mathcal{E}$ ,

$$\begin{array}{ccccc} \overline{X} & \xrightarrow{\quad} & 1 & & \\ \downarrow \overline{k} & & \downarrow \text{true} & & \\ Y & \xrightarrow{\quad} & \Omega & \xrightarrow{j} & \Omega \\ & \searrow \text{char}(k) & \downarrow \text{true} & & \\ & & \Omega & & \end{array} \quad (4)$$

(for more details, see [12]). One can construct  $\widetilde{k}$  in  $\mathcal{E}/B$ , similar to the above diagram.

Here, we proceed to improve [2, Vol. III, Proposition 9.2.5] as follows:

**Lemma 2.3.** *Let  $j$  be a topology in a topos  $\mathcal{E}$ . For every object  $B$  of  $\mathcal{E}$ , the pullback functor  $\Pi_B : \mathcal{E} \rightarrow \mathcal{E}/B$  preserves and reflects: denseness (closeness) and  $j$ -separated objects ( $j$ -sheaves).*

**Proof.** Let  $j$  be a topology on  $\mathcal{E}$  and  $B$  an object of  $\mathcal{E}$ . Preserving dense (closed) monomorphisms and sheaves (separated objects) in  $\mathcal{E}$  by the pullback functor  $\Pi_B$ , is standard and may be found in [2, Vol. III, Proposition 9.2.5]. To prove the rest of lemma, here we just show that  $\Pi_B$  reflects dense (closed) monomorphisms. To verify this claim, let  $g : A \rightarrow C$  be an arrow in  $\mathcal{E}$  for which  $\Pi_B(g)$  is a  $j_B$ -dense ( $j_B$ -closed) monomorphism. We show that  $g$  is  $j$ -dense ( $j$ -closed) monomorphism. As  $\Pi_B(g) = g \times \text{id}_B$  being monomorphism in  $\mathcal{E}/B$ , the arrow  $g$  is monomorphism in  $\mathcal{E}$  as well. For, let  $f, h$  in  $\mathcal{E}$  be two arrows such that  $gf = gh$ , we will have

$$\begin{aligned} gf = gh &\implies (g \times \text{id}_B)(f \times \text{id}_B) = (g \times \text{id}_B)(h \times \text{id}_B) \\ &\implies f \times \text{id}_B = h \times \text{id}_B \quad (g \times \text{id}_B \text{ is a monomorphism}) \\ &\implies f = h. \end{aligned}$$

Considering  $\overline{(\cdot)}$  and  $\widetilde{(\cdot)}$  as the universal closure operators corresponding to  $j$  and  $j_B$ , respectively. We get

$$\begin{aligned} \widetilde{\Pi_B(g)} &= \widetilde{g \times \text{id}_B} \\ &= \overline{g \times \text{id}_B} \quad (\text{by (3)}) \\ &= \overline{g} \times \text{id}_B, \end{aligned}$$

where the last equality is true since we have  $g \times \text{id}_B = (\pi_C^B)^{-1}(g)$ , and because of stability of universal closure operators under pullbacks we get  $\overline{(\pi_C^B)^{-1}(g)} = (\pi_C^B)^{-1}(\overline{g})$ . The above equalities imply that if  $\Pi_B(g)$  is  $j_B$ -dense ( $j_B$ -closed) monomorphism in  $\mathcal{E}/B$ , then  $g$  is  $j$ -dense ( $j$ -closed) monomorphism in  $\mathcal{E}$ .  $\square$

For any topology  $j$  on a topos  $\mathcal{E}$ , consider  $\mathcal{M}$  as the class of all  $j$ -dense monomorphisms in  $\mathcal{E}$ . Also, we denote by  $\mathcal{M}^\perp$  the class of all

arrows  $g : C \rightarrow D$  in  $\mathcal{E}$  such that for any  $f : A \rightarrow E$  in  $\mathcal{M}$  and every commutative square as in

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ f \downarrow & \nearrow w & \downarrow g \\ E & \xrightarrow{v} & D \end{array} \quad (5)$$

there exists a unique arrow  $w : E \rightarrow C$  in (5) such that the resulting triangles are commutative. In this case, we say that  $g$  is *right orthogonal* to  $f$ . Moreover, we say that the pair  $(\mathcal{M}, \mathcal{M}^\perp)$  forms a *factorization system* in  $\mathcal{E}$  if any arrow  $f$  in  $\mathcal{E}$  factors as  $f = me$ , where  $m \in \mathcal{M}$  and  $e \in \mathcal{M}^\perp$  (for more information, see [1]).

**Lemma 2.4.** *Let  $j$  be a topology on a topos  $\mathcal{E}$ . Then for each object  $B$  of  $\mathcal{E}$ , we have  $\mathcal{M}_B^\perp \subseteq \mathcal{M}^\perp$ , where  $\mathcal{M}_B$  is the class of all  $j_B$ -dense monomorphisms in  $\mathcal{E}/B$ .*

**Proof.** By Lemma 2.3 we get  $\mathcal{M}_B \subseteq \mathcal{M}$ . To reach the conclusion, let  $h : f \rightarrow g$  be an arrow in  $\mathcal{M}_B^\perp$ , where  $f : D \rightarrow B$  and  $g : E \rightarrow B$  are arrows in  $\mathcal{E}$ . Now, consider the commutative square

$$\begin{array}{ccc} A & \xrightarrow{u} & D \\ m \downarrow & & \downarrow h \\ C & \xrightarrow{v} & E \end{array} \quad (6)$$

where  $m : A \rightarrow C$  is in  $\mathcal{M}$ . Since by Lemma 2.3 the arrow  $m : fu \rightarrow gv$  in  $\mathcal{E}/B$  belongs to  $\mathcal{M}_B$  and  $h \in \mathcal{M}_B^\perp$ , there exists a unique arrow  $w : gv \rightarrow f$  in  $\mathcal{E}/B$  such that the following diagram commutes

$$\begin{array}{ccc} fu & \xrightarrow{u} & f \\ m \downarrow & \nearrow w & \downarrow h \\ gv & \xrightarrow{v} & g \end{array} \quad (7)$$

The arrow  $w : C \rightarrow D$  (as an arrow in  $\mathcal{E}$ ) which commutes the resulting triangles, is unique in the diagram (6). To prove this, let  $k : C \rightarrow D$  be an arrow in  $\mathcal{E}$  such that  $km = u$  and  $hk = v$ . Now, we have  $fk = (gh)k = gv$ , so  $k : gv \rightarrow f$  is an arrow in  $\mathcal{E}/B$  making all triangles in (7) commutative. Thus,  $k = w$  and the proof is complete.

□



**Definition 2.5.** Let  $j$  be a topology on a topos  $\mathcal{E}$ . We say that  $\mathcal{E}$  has enough  $j$ -sheaves if for every object  $A$  of  $\mathcal{E}$  there is a  $j$ -dense monomorphism  $A \rightarrowtail F$  where  $F$  is a  $j$ -sheaf.

Following [1] a class  $\mathcal{M}$  of morphisms in  $\mathcal{E}$  is a *left cancelable class* if  $gf \in \mathcal{M}$  implies  $f \in \mathcal{M}$ . In the following, we summarize the relation between left cancelable  $j$ -dense monomorphisms and factorization systems in a topos  $\mathcal{E}$  and its slices.

**Theorem 2.6.** Let  $j$  be a topology on a topos  $\mathcal{E}$ . Assume that for any object  $B$  of  $\mathcal{E}$ , the class  $\mathcal{M}_B$  of all  $j_B$ -dense monomorphisms in  $\mathcal{E}/B$  be left cancelable. Then the following are equivalent:

- (i) for any object  $B$  of  $\mathcal{E}$ ,  $(\mathcal{M}_B, \mathcal{M}_B^\perp)$  is a factorization system in  $\mathcal{E}/B$ ;
- (ii) for any object  $B$  of  $\mathcal{E}$ ,  $\mathcal{E}/B$  has enough  $j_B$ -sheaves;
- (iii) for any object  $B$  of  $\mathcal{E}$ , any object of  $\mathcal{E}/B$  is  $j_B$ -separated;
- (iv) for any object  $B$  of  $\mathcal{E}$ , any object of  $\mathcal{E}/B$  is  $j_B$ -sheaf;
- (v) any object of  $\mathcal{E}$  is  $j$ -sheaf;
- (vi) any object of  $\mathcal{E}$  is  $j$ -separated;
- (vii)  $\mathcal{E}$  has enough  $j$ -sheaves;
- (viii)  $(\mathcal{M}, \mathcal{M}^\perp)$  is a factorization system in  $\mathcal{E}$ .

**Proof.** That any  $j$ -sheaf is  $j$ -separated in  $\mathcal{E}$  yields that (v)  $\implies$  (vi) holds.

(vi)  $\implies$  (v). That any object of  $\mathcal{E}$  is  $j$ -separated it follows that  $\mathbf{Sep}_j(\mathcal{E})$  is the topos  $\mathcal{E}$  and then, every  $j$ -separated object is a  $j$ -sheaf as in [8, Theorem 2.1].

(iii)  $\implies$  (vi). Setting  $B = 1$ , then any object of  $\mathcal{E}$  is  $j$ -separated.

(vi)  $\implies$  (iii). The claim follows immediately from the fact that for any object  $B$  of  $\mathcal{E}$ ,

$$\mathbf{Sep}_{j_B}(\mathcal{E}/B) \cong \mathbf{Sep}_j(\mathcal{E})/B.$$

(see also [9]).

(viii)  $\implies$  (vii). By (viii), for any object  $A$  of  $\mathcal{E}$ , the unique arrow  $!_A : A \rightarrow 1$  factors as

$$\begin{array}{ccc} A & \xrightarrow{!_A} & 1 \\ & \searrow m & \nearrow !_C \\ & C & \end{array}$$

where  $!_C \in \mathcal{M}_1^\perp = \mathcal{M}^\perp$  and  $m \in \mathcal{M}_1 = \mathcal{M}$ . We remark that it is easy to check that for any object  $B$  of  $\mathcal{E}$ ,  $j_B$ -sheaves in  $\mathcal{E}/B$  are exactly the class of all objects of  $\mathcal{E}/B$  which belong to  $\mathcal{M}_B^\perp$ . Since  $!_C$  is an object in  $\mathcal{E}/1 = \mathcal{E}$  which is in  $\mathcal{M}_1^\perp$ , so  $!_C$  is a  $j_1$ -sheaf, or equivalently,  $C$  is a  $j$ -sheaf.

(vii)  $\implies$  (viii). Consider an arrow  $f : A \rightarrow B$  in  $\mathcal{E}$ . By using (vii), there exists a  $j$ -dense monomorphism  $\iota : A \rightarrowtail F$ , where  $F$  is a  $j$ -sheaf in  $\mathcal{E}$ . Now, we factor  $f$  as the composite arrow  $A \xrightarrow{(\iota, f)} F \times B \xrightarrow{\pi_B^F} B$ . Since  $\pi_F^B(\iota, f) = \iota \in \mathcal{M}$  and  $\mathcal{M}$  is a left cancelable class, so  $(\iota, f) \in \mathcal{M}$ . Also,  $F$  being  $j$ -sheaf, by Lemma 2.3 we have  $\pi_B^F$  is a  $j_B$ -sheaf in  $\mathcal{E}/B$ . By Lemma 2.4 we have  $\pi_B^F \in \mathcal{M}_B^\perp \subseteq \mathcal{M}^\perp$ , as required.

(vi)  $\implies$  (vii). First of all we know that any  $j$ -separated object of  $\mathcal{E}$  can be embedded into a  $j$ -sheaf (see, e.g. [12, Proposition V.3.4]). Let  $A$  be an object of  $\mathcal{E}$ . Then, by assumption  $A$  is  $j$ -separated, and there exists an embedding  $A \rightarrowtail F$ , where  $F$  is a  $j$ -sheaf. Now, take the closure of  $A$  in  $F$ . Since  $\overline{A}$  is closed in  $F$ , by [12, Lemma V.2.4], it is a  $j$ -sheaf. Since  $A$  is  $j$ -dense in  $\overline{A}$  we get the result.

(vii)  $\implies$  (vi). By assumption for any object  $A$  of  $\mathcal{E}$ , there is a  $j$ -dense monomorphism  $A \rightarrowtail F$  in  $\mathcal{E}$ , where  $F$  is a  $j$ -sheaf. Since any subobject of a  $j$ -sheaf is  $j$ -separated so  $A$  is  $j$ -separated.

For any object  $B$  of  $\mathcal{E}$ , setting  $\mathcal{E}/B$  instead of  $\mathcal{E}$  in (v), (vi), (vii) and (viii), we drive (i)  $\iff$  (ii)  $\iff$  (iii)  $\iff$  (iv).  $\square$

In the following, we will introduce two main classes of dense monomorphisms in a topos  $\mathcal{E}$ .

**Remark 2.7.** By diagram (4), one can easily obtain:

- (i) Let  $j = \text{id}_\Omega$  be the trivial topology on  $\mathcal{E}$ . Then  $j$ -dense monomorphisms are only the identity maps. Therefore, any object of  $\mathcal{E}$  is a  $j$ -sheaf. Also,  $j$ -closed monomorphisms are exactly all monomorphisms.
- (ii) Let  $j$  be the topology  $\text{true} \circ !_\Omega$  on  $\mathcal{E}$ , that is, the characteristic map of  $\text{id}_\Omega$ . Then,  $j$ -dense monomorphisms are exactly all monomorphisms. Furthermore,  $j$ -closed monomorphisms are just the identity maps.

Recall [1] that  $(\text{Mono}, \text{Mono}^\square)$  is a weak factorization system in any topos  $\mathcal{E}$ , where  $\text{Mono}$  is the class of all monomorphisms in  $\mathcal{E}$ . By Remark 2.7(ii), the class  $\text{Mono}$  is the class of all  $j$ -dense monomor-

phisms with respect to the topology  $j = \text{true} \circ !_\Omega$  on  $\mathcal{E}$ . Since the class  $\text{Mono}$  is left cancelable, so we can obtain a special case of Theorem 2.6 as follows. (Notice that by Lemma 2.3 for the topology  $j = \text{true} \circ !_\Omega$  and any object  $B$  of  $\mathcal{E}$ , the class  $\text{Mono}_B$  will be all monomorphisms in  $\mathcal{E}/B$ .)

**Corollary 2.8.** *For the topology  $j = \text{true} \circ !_\Omega$  on a topos  $\mathcal{E}$ , the following are equivalent:*

- (i) *for any object  $B$  of  $\mathcal{E}$ ,  $(\text{Mono}_B, \text{Mono}_B^\perp)$  is a factorization system in  $\mathcal{E}/B$ ;*
- (ii) *for any object  $B$  of  $\mathcal{E}$ ,  $\mathcal{E}/B$  has enough  $j_B$ -sheaves;*
- (iii) *for any object  $B$  of  $\mathcal{E}$ , any object of  $\mathcal{E}/B$  is  $j_B$ -sheaf;*
- (iv) *for any object  $B$  of  $\mathcal{E}$ , any object of  $\mathcal{E}/B$  is  $j_B$ -separated;*
- (v) *any object of  $\mathcal{E}$  is  $j$ -sheaf;*
- (vi) *any object of  $\mathcal{E}$  is  $j$ -separated;*
- (vii)  *$\mathcal{E}$  has enough  $j$ -sheaves;*
- (viii)  *$(\text{Mono}, \text{Mono}^\perp)$  is a factorization system in  $\mathcal{E}$ .*

### 3 Sheaves and sections of an arrow

In this section, among other things, we investigate a relationship between sheaves and sections of an arrow in a topos  $\mathcal{E}$ . We start to remind [3] that for any object  $B$  of  $\mathcal{E}$ , the pullback functor  $\Pi_B : \mathcal{E} \rightarrow \mathcal{E}/B$  has a right adjoint  $S : \mathcal{E}/B \rightarrow \mathcal{E}$  as for any  $f : X \rightarrow B$  we have the following pullback

$$\begin{array}{ccc} S(f) & \longrightarrow & 1 \\ \downarrow & & \downarrow i_B \\ X^B & \xrightarrow{f^B} & B^B \end{array} \quad (8)$$

where  $i_B$  is the transpose of  $\text{id}_B : 1 \times B \cong B \rightarrow B$  and  $f^B$  is the transpose of the composition arrow  $X^B \times B \xrightarrow{ev_X} X \xrightarrow{f} B$  by the exponential adjunction  $(-) \times B \dashv (-)^B$ ; that is,  $ev_B(i_B \times \text{id}_B) = \text{id}_B$  and  $ev_B(f^B \times \text{id}_B) = f \circ ev_X$ , where the natural transformation  $ev : (-)^B \times B \rightarrow (-)$  is the counit of the exponential adjunction. In fact,

in the Mitchell-Bénabou language, we can write

$$S(f) = \{h \mid (\forall c \in B) f \circ (h(c)) = c\}.$$

This means that we can call  $S(f)$  *the object of sections* of  $f$ .

Since any retract of an object in a topos (or in an arbitrary category) is an equalizer, so the topos  $\mathbf{Sh}_j(\mathcal{E})$  is closed under retracts. Furthermore, as  $\Pi_B \dashv S$ , by Lemma 2.3 we have that the pullback functor  $\Pi_B$  preserves dense monomorphisms, so  $S$  preserves sheaves (for details, see [9, Corollary 4.3.12]). (Roughly, for any object  $B \in \mathcal{E}$  and any adjoint  $F \dashv G : \mathcal{E} \rightarrow \mathcal{E}/B$  one can easily checked that the functor  $G$  preserves sheaves whenever  $F$  preserves dense monomorphisms.)

In the following theorem we will find a relationship between sheaves in  $\mathcal{E}/B$  and the object of sections of an arrow.

**Theorem 3.1.** *Let  $j$  be a topology on a topos  $\mathcal{E}$  and  $f : X \rightarrow B$  be an object of  $\mathcal{E}/B$ . Then,  $f$  is a  $j_B$ -sheaf in  $\mathcal{E}/B$ , whenever the graph of  $f$  which stands for the monomorphism  $(\text{id}_X, f) : f \rightarrow \pi_B^X$  in  $\mathcal{E}/B$ , is a section as well as  $S(f)$  is a  $j$ -sheaf in  $\mathcal{E}$ .*

**Proof.** We recall that in [3] it was proved if  $(\text{id}_X, f)$  is a section in  $\mathcal{E}/B$ , then  $f$  is a retract of  $\pi_B^{S(f)}$  in  $\mathcal{E}/B$ . As  $S(f)$  is a  $j$ -sheaf, by Lemma 2.3,  $\pi_B^{S(f)}$  is a  $j_B$ -sheaf in  $\mathcal{E}/B$ . But  $\mathbf{Sh}_{j_B}(\mathcal{E}/B)$  being closed under retracts, therefore  $f$  is a  $j_B$ -sheaf in  $\mathcal{E}/B$ .  $\square$

To the converse of Theorem 3.1, that the section functor  $S$  preserves sheaves it yields that if  $f : X \rightarrow B$  be a  $j_B$ -sheaf in  $\mathcal{E}/B$ , then  $S(f)$  is a  $j$ -sheaf in  $\mathcal{E}$ . Also, by Remark 2.7(ii), for  $j = \text{true} \circ !_\Omega$ , the monomorphism  $(\text{id}_X, f) : f \rightarrow \pi_B^X$  is  $j_B$ -dense in  $\mathcal{E}/B$  and then for a  $j_B$ -sheaf  $f : X \rightarrow B$ , it will be a section in  $\mathcal{E}/B$ .

In the rest of this section, for a small category  $\mathcal{C}$  we restrict our attention to obtain a version of Theorem 3.1 for injective presheaves in trivial slices of the presheaf topos  $\widehat{\mathcal{C}} = \mathbf{Sets}^{\mathcal{C}^{op}}$  which is close to the version over  $j$ -sheaves for the topology  $j = \text{true} \circ !_\Omega$  on  $\widehat{\mathcal{C}}$ . (See Proposition 3.5 below.) Note that the topology  $j = \text{true} \circ !_\Omega$  on  $\widehat{\mathcal{C}}$  is associated to the *chaotic or indiscrete Grothendieck topology* on  $\mathcal{C}$ . Recall [12] that in the presheaf topos  $\widehat{\mathcal{C}} = \mathbf{Sets}^{\mathcal{C}^{op}}$ , the exponential

object  $G^F$  is defined in each stage  $C$  of  $\mathcal{C}$  as  $G^F(C) = \text{Hom}_{\widehat{\mathcal{C}}}(Y(C) \times F, G)$ , where  $Y$  is the Yoneda embedding, that is

$$Y : \mathcal{C} \rightarrow \widehat{\mathcal{C}}; \quad Y(C) = \text{Hom}_{\mathcal{C}}(-, C).$$

Now, for an arrow  $\alpha : G \rightarrow F$  consider the arrows  $i_F : 1 \rightarrow F^F$  and  $\alpha^F : G^F \rightarrow F^F$  in  $\widehat{\mathcal{C}}$  as the transposes of  $\text{id}_F : 1 \times F \cong F \rightarrow F$  and  $\alpha \circ \text{ev}_G : G^F \times F \rightarrow F$ , respectively, by the exponential adjunction. We can observe

$$\forall C \in \mathcal{C}, \quad (i_F)_C : 1(C) = \{*\} \longrightarrow F^F(C); \quad (i_F)_C(*) = \pi_F^{Y(C)}. \quad (9)$$

Also, for any two objects  $C, D$  of  $\mathcal{C}$ , any  $\gamma$  in  $G^F(C)$  and any  $(k, y)$  in  $Y(C)(D) \times F(D)$  we have

$$(\alpha_C^F(\gamma))_D(k, y) = \alpha_D(\gamma_D(k, y)). \quad (10)$$

Remind that a presheaf  $G$  has a (unique) global section which means that in each stage  $C$  of  $\mathcal{C}$  there is a (unique) element  $\theta_C \in G(C)$  in such a way that for any arrow  $k : D \rightarrow C$  in  $\mathcal{C}$  we have

$$G(k)(\theta_C) = \theta_D. \quad (11)$$

Here, we find a special case that the exponential object and the object of sections in  $\widehat{\mathcal{C}}$  are exactly similar to **Sets**. First, we express some lemma required to achieve the goal.

**Lemma 3.2.** *Let  $j$  be the topology  $\text{true} \circ !_\Omega$  on  $\widehat{\mathcal{C}}$ . Then, the following assertions hold:*

- (i) *For any  $j$ -sheaf  $G$  in  $\widehat{\mathcal{C}}$ ,  $G$  has a unique global section. More generally, any injective presheaf  $G$  of  $\widehat{\mathcal{C}}$  has a global section.*
- (ii) *For any family  $\{G_\lambda\}_{\lambda \in \Lambda}$  in  $\widehat{\mathcal{C}}$ , the presheaf  $G = \prod_{\lambda \in \Lambda} G_\lambda$  is a  $j$ -sheaf (injective) in  $\widehat{\mathcal{C}}$  iff for all  $\lambda \in \Lambda$ ,  $G_\lambda$  is a  $j$ -sheaf (injective) in  $\widehat{\mathcal{C}}$ .*

**Proof.** (i) Let  $G$  be a  $j$ -sheaf in  $\widehat{\mathcal{C}}$  and consider the coproduct object  $G \sqcup 1$  in  $\widehat{\mathcal{C}}$ . By Remark 2.7(ii), there exists a unique natural transformation  $\eta : G \sqcup 1 \rightarrow G$  in  $\widehat{\mathcal{C}}$  such that the following diagram

commutes (if  $G$  being injective, the arrow  $\eta$  is not necessarily unique)

$$\begin{array}{ccc} G & \xrightarrow{\text{id}_G} & G \\ \downarrow \iota & \nearrow \eta & \\ G \sqcup 1 & & \end{array}$$

where  $\iota : G \rightarrow G \sqcup 1$  is the injection arrow. Now, we will denote  $\eta_C(*)$  by an element  $\theta_C$  in  $G(C)$  in each stage  $C$  of  $\mathcal{C}$ . Since  $\eta : G \sqcup 1 \rightarrow G$  is natural, so for any arrow  $k : D \rightarrow C$  in  $\mathcal{C}$  the following square commutes

$$\begin{array}{ccc} (G \sqcup 1)(D) & \xrightarrow{\eta_D} & G(D) \\ (G \sqcup 1)(k) \uparrow & & \uparrow G(k) \\ (G \sqcup 1)(C) & \xrightarrow{\eta_C} & G(C) \end{array}$$

Then, we have

$$\begin{aligned} G(k)(\theta_C) &= G(k)(\eta_C(*)) \\ &= \eta_D((G \sqcup 1)(k)(*)) \\ &= \eta_D(1(k)(*)) = \theta_D. \end{aligned}$$

This is the required result.

(ii) *Necessity.* Let  $G$  be a  $j$ -sheaf (injective) in  $\widehat{\mathcal{C}}$ . For any  $\lambda, \mu \in \Lambda$ , we define  $\alpha^{\lambda\mu} : G_\lambda \rightarrow G_\mu$  such that in each stage  $C$  of  $\mathcal{C}$  and for each  $x \in G_\lambda(C)$ , we have  $\alpha_C^{\lambda\mu}(x) = \theta_C^\mu$ , where  $\theta_C^\mu$  is the  $\mu$ -th component of  $\theta_C$  corresponding to  $G$  in (i). Now, we will show that for any  $\lambda, \mu \in \Lambda$ ,  $\alpha^{\lambda\mu}$  is a natural transformation in  $\widehat{\mathcal{C}}$ , that is for any arrow  $k : D \rightarrow C$  in  $\mathcal{C}$  the following diagram is commutative

$$\begin{array}{ccc} G_\lambda(D) & \xrightarrow{\alpha_D^{\lambda\mu}} & G_\mu(D) \\ G_\lambda(k) \uparrow & & \uparrow G_\mu(k) \\ G_\lambda(C) & \xrightarrow{\alpha_C^{\lambda\mu}} & G_\mu(C) \end{array}$$

For, consider an element  $x \in G_\lambda(C)$  we get

$$\begin{aligned} G_\mu(k)(\alpha_C^{\lambda\mu}(x)) &= G_\mu(k)(\theta_C^\mu) \\ &= \theta_D^\mu && \text{(by (11))} \\ &= \alpha_D^{\lambda\mu}(G_\lambda(k)(x)). \end{aligned}$$

Now, for any  $\lambda \in \Lambda$ , consider the family  $\{\gamma_\mu : G_\lambda \rightarrow G_\mu\}_{\mu \in \Lambda}$  in  $\widehat{\mathcal{C}}$  such that for each  $\lambda \neq \mu \in \Lambda$  we have  $\gamma_\mu = \alpha^{\lambda\mu}$  and  $\gamma_\lambda = \text{id}_{G_\lambda}$ . Since  $G$

is the product  $\prod_{\lambda \in \Lambda} G_\lambda$ , so there is a unique natural transformation  $\gamma : G_\lambda \rightarrow G$  such that  $p_\mu \gamma = \gamma_\mu$  and  $p_\lambda \gamma = \text{id}_{G_\lambda}$ , for all  $\lambda, \mu \in \Lambda$  and the projections  $p_\lambda$ . Thus, for any  $\lambda \in \Lambda$ ,  $G_\lambda$  is a retract of the  $j$ -sheaf (injective)  $G$  and then,  $G_\lambda$  is a  $j$ -sheaf (injective).

*Sufficiency.* By the universal property of the product presheaf  $G$ , the unique arrow in the definition of a sheaf easily follows.  $\square$

We recall [12] that in each stage  $C$  of  $\mathcal{C}$  the object  $\Omega(C)$  of  $\widehat{\mathcal{C}}$  is the set of all sieves on  $C$ . Also, the arrow  $\text{true}_C : 1(C) = \{*\} \rightarrow \Omega(C)$  assigns to  $*$ , the *maximal sieve*  $t(C)$  of  $\Omega(C)$ , that is all arrows with codomain  $C$  of  $\mathcal{C}$ .

**Remark 3.3.** Note that the topology  $j = \text{true} \circ !_\Omega$  on  $\widehat{\mathcal{C}}$  is the unique topology on  $\widehat{\mathcal{C}}$  that satisfies Lemma 3.2. To show this, for a  $j$ -sheaf  $G$  of  $\widehat{\mathcal{C}}$ , consider the injection  $\iota : G \rightarrow G \sqcup 1$  in  $\widehat{\mathcal{C}}$ . In each stage  $C$  of  $\mathcal{C}$  we have  $\text{char}(\iota)_C(*) = \emptyset$ . Now, let  $j$  be a topology on  $\widehat{\mathcal{C}}$ . If  $\iota$  is  $j$ -dense monomorphism, then in each stage  $C$  of  $\mathcal{C}$  we have  $j_C(\emptyset) = t(C)$ . Now, for any sieve  $S \in \Omega(C)$  by Definition 1.1 we get

$$\begin{aligned} t(C) &= j_C(\emptyset) = j_C(\emptyset \cap S) \\ &= j_C(\emptyset) \cap j_C(S) = t(C) \cap j_C(S) = j_C(S). \end{aligned}$$

Thus,  $j_C$  is the constant function on  $t(C)$ , as required.

Let  $F$  be the constant presheaf on a set  $A$ . One can easily checked that the exponential adjunction  $(-) \times F \dashv (-)^F$  is determined by, for any presheaf  $G$  in  $\widehat{\mathcal{C}}$ , the exponential presheaf  $G^F$  assigns to any object  $C$  of  $\mathcal{C}$ , the hom-set  $\text{Hom}_{\mathbf{Sets}}(A, G(C))$  and to any arrow  $f : C \rightarrow D$  of  $\mathcal{C}$ , the function

$$G^F(f) : \text{Hom}_{\mathbf{Sets}}(A, G(D)) \longrightarrow \text{Hom}_{\mathbf{Sets}}(A, G(C))$$

given by  $G^F(f)(g) = G(f) \circ g$ . As any function  $f : A \rightarrow G(C)$  can be considered as a sequence  $(x_a)_{a \in A} \in \prod_A G(C)$ , it yields that one has

$$\forall C \in \mathcal{C}, \quad G^F(C) \cong \prod_A G(C). \quad (12)$$

By (12), (9) and (10), it is convenient to see that for each arrow  $\alpha : G \rightarrow F$  in  $\widehat{\mathcal{C}}$  in which  $F$  stands for the constant presheaf on a set  $A$ ,

we get

$$\forall C \in \mathcal{C}, \quad S(\alpha)(C) \cong \prod_{a \in A} \alpha_C^{-1}(a). \quad (13)$$

Now, we will extract a special case of Theorem 3.1 in  $\widehat{\mathcal{C}}$ . First, let  $\alpha : G \rightarrow F$  be an arrow in  $\widehat{\mathcal{C}}$  in which  $F$  is the constant presheaf on a set  $A$ . For each element  $a$  of  $A$ , consider the subpresheaf  $H_a$  of  $G$  such that  $H_a(C) = \alpha_C^{-1}(a)$ , for any object  $C$  of  $\mathcal{C}$ . Since limits in  $\widehat{\mathcal{C}}$  are constructed pointwise, so (13) shows that  $S(\alpha) \cong \prod_{a \in A} H_a$ .

**Proposition 3.4.** *Let  $j$  be the topology  $\text{true} \circ !_\Omega$  on  $\widehat{\mathcal{C}}$  and  $\alpha : G \rightarrow F$  an arrow in  $\widehat{\mathcal{C}}$ , where  $F$  is the constant presheaf on a set  $A$ . Then,  $\alpha$  is a  $j_F$ -sheaf in  $\widehat{\mathcal{C}}/F$  iff the monomorphism  $(\text{id}_G, \alpha) : \alpha \mapsto \pi_F^G$  is a section in  $\widehat{\mathcal{C}}/F$  as well as for any  $a \in A$ , the subpresheaf  $H_a$  of  $G$  is a  $j$ -sheaf in  $\widehat{\mathcal{C}}$ .*

**Proof.** We deduce the result by Theorem 3.1, Lemma 3.2(ii) and (13).  $\square$

Since in topoi regular monomorphisms are exactly monomorphisms, so by [3, Theorem 1.2], Lemma 3.2(ii) and (13), the following now gives which we are interested in.

**Proposition 3.5.** *Let  $\alpha : G \rightarrow F$  be an arrow in  $\widehat{\mathcal{C}}$ , where  $F$  is the constant presheaf on a set  $A$ . Then,  $\alpha$  is injective in  $\widehat{\mathcal{C}}/F$  iff the monomorphism  $(\text{id}_G, \alpha) : \alpha \mapsto \pi_F^G$  is a section in  $\widehat{\mathcal{C}}/F$  as well as for any  $a \in A$ , the subpresheaf  $H_a$  of  $G$  is injective.*

In the case when  $\mathcal{C}$  is a monoid, we obtain

**Example 3.6.** Let  $M$  be a monoid and  $M\text{-Sets}$  the topos of all (right) representations of a fixed monoid  $M$ . Since  $M$  is a small category with just one object, for two  $M$ -sets  $X, B$  we have  $X^B = \text{Hom}_{M\text{-Sets}}(M \times B, X)$ , where  $M \times B$  has the componentwise action. Hence, by (9) and (10), for any equivariant map  $f : X \rightarrow B$ , in the diagram (8) we observe

$$i_B(*) = \pi_B^M : M \times B \rightarrow B, \quad (14)$$

and

$$\forall h \in X^B, \quad \forall (m, b) \in M \times B, \quad (f^B(h))(m, b) = fh(m, b). \quad (15)$$



Note that one writes any equivariant map  $h : M \times B \rightarrow X$  in  $X^B$  as a sequence  $((x_{m,b})_{b \in B})_{m \in M}$ , consisting of elements  $x_{m,b} = h(m, b)$  of  $X$ , for any  $(m, b) \in M \times B$ . Also,  $h$  being equivariant map means that

$$\forall n, m \in M, \forall b \in B, \quad x_{mn, bn} = x_{m, b}n.$$

Hence, we obtain that  $X^B$  is equal to

$$\{((x_{m,b})_b)_m \in \prod_{m \in M} \prod_{b \in B} X \mid \forall n, m \in M, \forall b \in B, x_{mn, bn} = x_{m, b}n\}. \quad (16)$$

Now, by (8), (14) and (15) we have

$$\begin{aligned} S(f) &= \{((x_{m,b})_b)_m \in X^B \mid f^B(((x_{m,b})_b)_m) = \pi_B^M = ((b)_b)_m\} \\ &= \{((x_{m,b})_b)_m \in X^B \mid ((f(x_{m,b}))_b)_m = ((b)_b)_m\} \\ &= \{((x_{m,b})_b)_m \in X^B \mid \forall m \in M, \forall b \in B, x_{m,b} \in f^{-1}(b)\}, \end{aligned}$$

Hence, by (16) we interpret a simple form of underlying set of the  $M$ -set  $S(f)$  in the topos  $M\text{-}\mathbf{Sets}$  as follows

$$\{((x_{m,b})_b)_m \in \prod_{m \in M} \prod_{b \in B} f^{-1}(b) \mid \forall n, m \in M, \forall b \in B, x_{mn, bn} = x_{m, b}n\}.$$

If  $B$  has the trivial action  $\cdot$ , that is  $\cdot = \pi_1 : B \times M \rightarrow B$  the first projection, then by (12) and (13) we can obtain  $X^B \cong \prod_B X$  and  $S(f) \cong \prod_{b \in B} f^{-1}(b)$ .

Furthermore, recall [12] that for a group  $G$  and two  $G$ -sets  $X, B$ , we have

$$X^B = \{h : B \rightarrow X \mid h \text{ is a function}\} \cong \prod_B X \quad (17)$$

as two sets. According to the action on  $X^B$ , under the isomorphism (17), the action on  $\prod_B X$  is given by  $(x_b)_{b \in B} \cdot g = (x_{bg^{-1}} \cdot g)_{b \in B}$ , for any  $g \in G$  and  $(x_b)_{b \in B} \in \prod_B X$ . Also, by (17) for any equivariant map  $f : X \rightarrow B$  in  $G\text{-}\mathbf{Sets}$ , in a similar way to (13), we have  $S(f) \cong \prod_{b \in B} f^{-1}(b)$ .

## 4 $j$ -essential extensions in a topos

This section is devoted to introduce a class of monomorphisms in an elementary topos, which we call these ‘ $j$ -essential monomorphisms’.

We present some equivalent forms of these and some their properties. Meanwhile, we prove that any presheaf in a presheaf topos has a maximal essential extension.

Remind that a monomorphism  $\iota : A \rightarrowtail B$  is called *essential* whenever for each arrow  $g : B \rightarrow C$  such that  $g\iota$  is a monomorphism, then  $g$  is a monomorphism also. Now, we define a  $j$ -essential monomorphism in a topos  $\mathcal{E}$  as follows.

**Definition 4.1.** For a topology  $j$  on  $\mathcal{E}$ , a monomorphism  $\iota : A \rightarrowtail B$  is called  *$j$ -essential* whenever it is  $j$ -dense as well as essential. In this case, we say that  $B$  is a  *$j$ -essential extension* of  $A$  and we write  $A \subseteq_j B$ .

We shall say an arrow  $f : A \rightarrow B$  in  $\mathcal{E}$  is  *$j$ -dense* whenever the subobject  $f(A)$ , which is the image of  $f$ , is  $j$ -dense in  $B$ . In this way, any epimorphism in  $\mathcal{E}$  becomes  $j$ -dense. ( For the definition of image of an arrow in a topos, see [12].)

The following gives some equivalent definitions of  $j$ -essential monomorphisms in a topos  $\mathcal{E}$ .

**Lemma 4.2.** *Let  $j$  be a topology on  $\mathcal{E}$  and  $\iota : A \rightarrowtail B$  a  $j$ -dense monomorphism. Then, the following are equivalent:*

- (i) *for any  $g : B \rightarrow C$ ,  $g$  is a monomorphism whenever  $g\iota$  is a monomorphism;*
- (ii) *for any  $g : B \rightarrow C$ ,  $g$  is a  $j$ -dense monomorphism whenever  $g\iota$  is a  $j$ -dense monomorphism;*
- (iii) *for any  $g : B \rightarrow C$ ,  $g$  is a monomorphism whenever  $g\iota$  is a  $j$ -dense monomorphism.*

**Proof.** (i)  $\implies$  (ii) and (iii)  $\implies$  (ii) are proved by [9, A.4.5.11(iii)].  
(ii)  $\implies$  (iii) is clear.  
(ii)  $\implies$  (i). Consider an arrow  $g : B \rightarrow C$  for which  $g\iota$  is monomorphism. We show that  $g$  is monomorphism also. Assume that  $B \xrightarrow{k} g(B) \xrightarrow{m} C$  is the image factorization of the arrow  $g$ . Since  $g\iota = m(k\iota)$  and  $g\iota$  is monomorphism, it follows that the arrow  $k\iota$  is a monomor-

phism. Meanwhile, we get

$$\begin{aligned}
g(B) &= k(B) && (\text{as } k \text{ is epic}) \\
&= k(\overline{A}) && (\text{as } \iota \text{ is dense}) \\
&\subseteq \overline{k(A)} \\
&\subseteq g(B).
\end{aligned}$$

Therefore,  $g(B) = \overline{k(A)} = \overline{k\iota(A)}$ . It follows that the compound monomorphism  $k\iota : A \rightarrowtail g(B)$  is dense monomorphism and by (ii),  $k$  is also. That  $k$  is monomorphism and so isomorphism, yields that  $g$  is monomorphism.  $\square$

We point out that the proof of (ii)  $\implies$  (i) of Lemma 4.2 shows that any composite  $k\iota$ , for an epic  $k$  and a dense monomorphism  $\iota$ , is dense.

The following shows that  $j$ -essential monomorphisms in  $\mathcal{E}$  are closed under composition.

**Proposition 4.3.** *Let  $j$  be a topology on  $\mathcal{E}$ . For two subobjects  $A \xrightarrow{\iota} A' \xrightarrow{\iota'} B$  in  $\mathcal{E}$ , then  $A \subseteq_j B$  iff  $A \subseteq_j A'$  and  $A' \subseteq_j B$ .*

**Proof.** By [9, 13, A.4.5.11(iii)], one has  $\iota'\iota$  is  $j$ -dense iff  $\iota'$  and  $\iota$  are  $j$ -dense.

*Necessity.* First, by Lemma 4.2(i), we show that  $A \subseteq_j A'$ . To do so, consider an arrow  $f' : A' \rightarrow C$  for which  $f'\iota$  is a monomorphism. Now, by [12, Corollary IV. 10. 3], the object  $C$  can be embedded into an injective object  $D$  as in  $C \xrightarrow{\nu} D$  and hence there is an arrow  $\tilde{f}' : B \rightarrow D$  such that  $\tilde{f}'\iota' = \nu f'$ . Since  $A \subseteq_j B$  and  $\tilde{f}'\iota'\iota = \nu f'\iota$  is a monomorphism, we deduce that  $\tilde{f}'$  is a monomorphism. As  $\tilde{f}'\iota' = \nu f'$  it follows that  $f'$  is a monomorphism.

To prove  $A' \subseteq_j B$ , choose an arrow  $f : B \rightarrow C$  for which  $f\iota'$  is a monomorphism. Then,  $f\iota'\iota$  is also a monomorphism. Now  $A \subseteq_j B$  implies that  $f$  is a monomorphism, as required.

*Sufficiency.* Let  $f : B \rightarrow C$  be an arrow in  $\mathcal{E}$  such that  $f\iota'\iota$  is a monomorphism. Since  $A \subseteq_j A'$  and  $(f\iota')\iota = f\iota'\iota$  is a monomorphism, it concludes that  $f\iota'$  is a monomorphism. Using  $A' \subseteq_j B$ , we achieve that  $f$  is a monomorphism and hence  $A \subseteq_j B$ .  $\square$

In the following, we achieve another property of  $j$ -essential monomorphisms in  $\mathcal{E}$ .

**Lemma 4.4.** *Let  $j$  be a topology on  $\mathcal{E}$ . If  $A \subseteq_j B$  and  $A$  is embedded in a  $j$ -sheaf  $F$ , then  $B$  also is embedded in  $F$ .*

**Proof.** Let  $\iota : A \rightarrowtail B$  be a  $j$ -essential monomorphism and  $m : A \rightarrowtail F$  an arbitrary embedding. Since  $F$  is a  $j$ -sheaf, there exists a unique morphism  $f : B \rightarrow F$  making the diagram below commutative;

$$\begin{array}{ccc} A & \xrightarrow{m} & F \\ \downarrow \iota & \nearrow f & \\ B & & \end{array}$$

As  $A \subseteq_j B$  being  $j$ -essential,  $f$  is an embedding, as required.  $\square$

By Remark 2.7(ii), essential monomorphisms in a topos  $\mathcal{E}$  are exactly  $j$ -essential monomorphisms in  $\mathcal{E}$  with respect to the topology  $j = \text{true} \circ !_\Omega$  on  $\mathcal{E}$ .

Now, we would like to prove that any presheaf in  $\widehat{\mathcal{C}}$  has a maximal essential extension.

**Theorem 4.5.** *Any presheaf in  $\widehat{\mathcal{C}}$  has a maximal essential extension.*

**Proof.** Let  $F$  be a presheaf in  $\widehat{\mathcal{C}}$  and  $G$  an injective presheaf into which  $F$  can be embedded. By Lemma 4.4, we can assume that both  $F$  and all its essential extensions are subpresheaves of  $G$ . Consider  $\sum$  as the set of all essential extensions of  $F$  which is a poset under subpresheaf inclusion  $\subseteq$ . Since the arrow  $\text{id}_F$  is an essential extension of  $F$ , it follows that  $\sum$  is non-empty. If

$$\dots \subseteq F_i \subseteq \dots,$$

$i \in I$ , is a chain in  $\sum$ , then the subpresheaf  $H$  of  $G$  given by  $H(C) = \bigcup_{i \in I} F_i(C)$  for any object  $C$  in  $\mathcal{C}$  is an upper bound of this chain. Now we show that  $H$  lies in  $\sum$ , i.e.,  $H$  is an essential extension of  $F$ . To achieve this, let  $\alpha : H \rightarrow K$  be an arrow in  $\widehat{\mathcal{C}}$  such that the restriction arrow  $\alpha|_F$  is a monomorphism. We prove that  $\alpha$  is a monomorphism. To verify this claim, we show that for any  $C \in \widehat{\mathcal{C}}$ , the function  $\alpha_C : \bigcup_{i \in I} F_i(C) \rightarrow K(C)$  is one to one. Take  $a, b \in \bigcup_{i \in I} F_i(C)$ ,  $a \neq b$ . Then there is a  $j \in I$  such that  $a, b \in F_j(C)$ . Denote  $\alpha|_{F_j}$  by  $\alpha_j$ . Since  $F_j$

is an essential extension of  $F$  and  $\alpha_j|_F = \alpha|_F$ , it implies that  $\alpha_j$  is a monomorphism. Now

$$\alpha_C(a) = (\alpha_j)_C(a) \neq (\alpha_j)_C(b) = \alpha_C(b).$$

Therefore,  $\alpha$  is a monomorphism. Thus,  $H \in \Sigma$ . Now it follows from Zorn's Lemma that there is a maximal element  $M$  in  $\Sigma$ . Then,  $M$  is a maximal essential extension of  $F$ .  $\square$

It is straightforward to see that any essential extension of  $B$  can be embedded in any injective extension of  $B$ .

For a topology  $j$  on a topos  $\mathcal{E}$ , by a *j-injective object* we mean an injective object with respect to the class of all  $j$ -dense monomorphisms in  $\mathcal{E}$ .

The following shows that the  $j$ -injective presheaves ( $j$ -sheaves) in  $\widehat{\mathcal{C}}$  have no proper  $j$ -essential extension.

**Proposition 4.6.** *Let  $j$  be a topology on  $\widehat{\mathcal{C}}$  and  $F$  a  $j$ -injective presheaf ( $j$ -sheaf) in  $\widehat{\mathcal{C}}$ . Then,  $F$  has no proper  $j$ -essential extension.*

**Proof.** Suppose that  $G$  is a proper  $j$ -essential extension of  $F$  and so  $F$  is a  $j$ -dense subpresheaf of  $G$  and  $F \neq G$ . Thus there is an object  $C$  of  $\mathcal{C}$  such that  $G(C) \not\subseteq F(C)$  and then, an  $a \in G(C)$  such that  $a \notin F(C)$ . Since  $F$  is  $j$ -injective ( $j$ -sheaf) implies that there is an arrow  $\alpha : G \rightarrow F$  for which  $\alpha|_F = \text{id}_F$ . That  $a \notin F(C)$  and  $\alpha_C(a) \in F(C)$  follows that  $a \neq \alpha_C(a)$ . But  $\alpha_C(\alpha_C(a)) = \alpha_C(a)$ . Then,  $\alpha_C$  and so  $\alpha$  is not a monomorphism although  $\alpha|_F = \text{id}_F$  is. This shows that  $G$  is not a proper  $j$ -essential extension of  $F$  and it is a contradiction.  $\square$

The following shows that the pullback functor  $\Pi_B$  reflects  $j$ -essential extensions.

**Proposition 4.7.** *Let  $j$  be a topology in a topos  $\mathcal{E}$ . For every object  $B \in \mathcal{E}$ , the pullback functor  $\Pi_B : \mathcal{E} \rightarrow \mathcal{E}/B$  reflects  $j$ -essential monomorphisms.*

**Proof.** Let  $f : A \rightarrow C$  be an arrow in  $\mathcal{E}$  such that  $\Pi_B(f)$  is a  $j_B$ -essential monomorphism. We show that  $f$  is a  $j$ -essential monomorphism. By Lemma 2.3,  $f$  is a  $j$ -dense monomorphism in  $\mathcal{E}$ . Let  $g : C \rightarrow D$  be an arrow in  $\mathcal{E}$  such that  $gf$  is a monomorphism.

We show that  $g$  is too. Since  $gf$  is a monomorphism, the arrow  $(g \times \text{id}_B)\Pi_B(f) = (gf) \times \text{id}_B$  is also a monomorphism. As  $\Pi_B(f)$  is  $j_B$ -essential, so  $g \times \text{id}_B$  is a monomorphism. Then  $g$  is a monomorphism. This is the required result.  $\square$

Recall [10] that a *weak topology* on a topos  $\mathcal{E}$  is a morphism  $j : \Omega \rightarrow \Omega$  such that:

- (i)  $j \circ \text{true} = \text{true}$ ;
- (ii)  $j \circ \wedge \leq \wedge \circ (j \times j)$ , in which  $\leq$  stands for the internal order on  $\Omega$ . Meanwhile, a weak topology  $j$  on  $\mathcal{E}$  is said to be *productive* if  $j \circ \wedge = \wedge \circ (j \times j)$ .

In what follows, we review the whole paper for a weak topology  $j$  on a topos  $\mathcal{E}$  instead of a topology.

**Remark 4.8.** Similar to [9, A.4.5.11(ii)], one can easily check that for a weak topology  $j$  on  $\mathcal{E}$  pushouts also preserve dense monomorphisms. Hence, we can obtain a version of Lemma 2.1 for a weak topology  $j$  on  $\mathcal{E}$  as well. One can observe that completely analogous assertions to Lemmas 4.2, 4.4 and 2.3, Proposition 4.6 and Theorem 3.1, hold for a weak topology  $j$  on  $\mathcal{E}$ . But, by [10], in the proof of Theorem 2.6, the part (vi)  $\implies$  (vii) is true for a productive weak topology  $j$  on  $\mathcal{E}$ . The rest parts of this proof satisfies for weak topologies.

Recall [10] that, for a weak topology  $j$  on  $\mathcal{E}$ , it is convenient to see that if the composite subobject  $mn$  is dense then so are  $m$  and  $n$ . In contrast with topologies [9, A.4.5.11(iii)], the converse is not necessarily true. Hence, the sufficiency part of Proposition 4.3 does not necessarily hold for a weak topology  $j$  on a topos  $\mathcal{E}$ . The necessity part of this proposition satisfies for a weak topology  $j$  as well.

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